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Gravitational Cheshire effect: Nonminimally coupled scalar fields may not curve spacetime

Eloy Ayón–Beato[1,2], **Cristián Martínez**[1], **Ricardo Troncoso**[1] **and Jorge Zanelli**[1] *

[1] *Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.*
[2] *Departamento de Física, CINVESTAV–IPN, Apdo. Postal 14–740, 07000, México D.F., México.*

Abstract: It is shown that flat spacetime can be dressed with a real scalar field that satisfies the nonlinear Klein-Gordon equation without curving spacetime. Surprisingly, this possibility arises from the nonminimal coupling of the scalar field with the curvature, since a footprint of the coupling remains in the energy-momentum tensor even when gravity is switched off. Requiring the existence of solutions with vanishing energy-momentum tensor fixes the self-interaction potential as a local function of the scalar field depending on two coupling constants. The solutions describe shock waves and, in the Euclidean continuation, instanton configurations in any dimension. As a consequence of this effect, the tachyonic solutions of the free massive Klein-Gordon equation become part of the vacuum.

* *E-mail:* ayon-at-cecs.cl, martinez-at-cecs.cl, ratron-at-cecs.cl, jz-at-cecs.cl

1. Introduction

According to Einstein's theory of gravitation, gravity is the manifestation of the curvature of spacetime produced by the presence of matter. In the absence of sources, the maximally symmetric solution of the Einstein equations —without cosmological constant— is Minkowski spacetime. The converse, however, is not necessarily true: Minkowski space does not imply that spacetime is devoid of matter. The purpose of this article is to construct a simple and explicit example where this possibility is realized.

One of the simplest sources is provided by a real scalar field and the phenomenon described above occurs if the field is nonminimally coupled to gravity. What is remarkable about this possibility is that if the gravitational field is switched off, the presence of the nonminimal coupling persists. That is, if the background geometry is flat, so that the coupling term in the action vanishes, the footprint of the coupling to gravity is still present in the energy-momentum tensor. In this sense, the gravitational field plays the role of Alice's Cheshire cat, where the coupling constant is the grin [1].

A scalar field nonminimally coupled to gravity in D dimensions is described by the action

$$I = \int \mathrm{d}^D x\sqrt{-g}\left(\frac{1}{2\kappa}R \ + \frac{1}{2}\phi\Box\phi - \frac{1}{2}\zeta\, R\,\phi^2 - V(\phi)\right). \tag{1.1}$$

The field equations in flat spacetime reduce to

$$\Box\phi = V'(\phi)\ , \tag{1.2}$$

and

$$\Theta_{\mu\nu} \equiv T_{\mu\nu} + \zeta(\eta_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\phi^2 = 0\ , \tag{1.3}$$

where $T_{\mu\nu}$ is the standard energy-momentum tensor for a minimally coupled scalar field,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\right). \tag{1.4}$$

It was observed long ago, that adding an identically conserved symmetric tensor to (1.4) leads to a theory with improved renormalizability properties in flat space [2, 3]. The additional term in Eq. (1.3) is a straightforward generalization of the "improved" energy-momentum tensor considered there.

For the minimally coupled case $\zeta = 0$, Eq. (1.3) reduces to $T_{\mu\nu} = 0$. Any solution of (1.2) with $T_{\mu\nu} = 0$ is necessarily a constant scalar field ϕ_0 such that $V(\phi_0) = V'(\phi_0) = 0$. Thus, in this case, the solutions that do not generate a back reaction are trivial.

In the general case, under the same asymptotic conditions for the scalar field in flat space as for the conventional tensor (1.4), $\Theta_{\mu\nu}$ gives the same charge generators for the Poincaré group. However, the improvement allows for weaker asymptotic conditions, which include a wider class of solutions with finite charges. In particular, the effect of relaxing the asymptotic conditions gives rise to the existence of nontrivial solutions of the nonlinear Klein-Gordon equation with vanishing $\Theta_{\mu\nu}$ for any nonzero value of ζ.

As it is shown below, requiring the existence of nontrivial solutions of $\Theta_{\mu\nu} = 0$, fixes the self-interaction potential $V(\phi)$ as a local function of the scalar field depending on two coupling constants, given by[1]

$$V(\phi) = \frac{2\zeta^2}{(1-4\zeta)^2}\left(\lambda_1 \phi^{\frac{1-2\zeta}{\zeta}} + 8(D-1)(\zeta-\zeta_D)\lambda_2 \phi^{\frac{1}{2\zeta}}\right), \tag{1.5}$$

where $\zeta_D := \frac{D-2}{4(D-1)}$. Note that if $\zeta = \zeta_D$, matter becomes conformally coupled and the potential has a single coupling constant,

$$V(\phi) = \frac{(D-2)^2}{8}\lambda_1 \phi^{\frac{2D}{D-2}}\ , \tag{1.6}$$

and $\Theta_{\mu\nu}$ becomes traceless, reflecting the scale invariance of the scalar field equation.

The scalar field configurations satisfying Eq. (1.3) couple to the gravitational field without curving spacetime. They are found to describe shock waves and, in the Euclidean continuation, they correspond to instanton configurations.

2. Solutions

Note that Eq. (1.2) is a consequence of the conservation of the energy-momentum tensor provided $\nabla_\mu \phi \neq 0$. Therefore, solving Eq. (1.3) for a nonconstant scalar field is sufficient to satisfy Eq. (1.2). In standard Minkowski coordinates, $x^\mu = (t, x^i)$, $i = 1, \ldots, D-1$, these equations can be written as

$$0{=}\Theta_{\mu\nu} = \frac{(2\zeta)^2}{1-4\zeta}\frac{\phi^2}{\sigma}\partial^2_{\mu\nu}\sigma, \qquad \mu \neq \nu, \tag{2.1a}$$

$$0{=}\Theta_{ii} + \Theta_{tt} {=} \frac{(2\zeta)^2}{1-4\zeta}\frac{\phi^2}{\sigma}\left(\partial^2_{ii}\sigma + \partial^2_{tt}\sigma\right) \text{ (no sum)}, \tag{2.1b}$$

$$0{=}\Theta_{tt} {=} V(\phi) {-} \frac{(2\zeta)^2\phi^2}{(1-4\zeta)\sigma}\left[\frac{\partial_\alpha\sigma\partial^\alpha\sigma}{2(1-4\zeta)\sigma} {-} \sum_{i=1}^{D-1}\partial^2_{ii}\sigma\right], \tag{2.1c}$$

where we have defined

$$\sigma = \phi^{(4\zeta-1)/2\zeta}\ , \quad (\zeta \neq 0,\ \ \zeta \neq 1/4). \tag{2.2}$$

It follows from Eq. (2.1a) that σ is completely separable,

$$\sigma = T(t) + X^1(x^1) + \cdots + X^{D-1}(x^{D-1})\ . \tag{2.3}$$

On the other hand, combining Eqs. (2.1b), one obtains

$$-\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 X^1}{\mathrm{d}(x^1)^2} = \ldots = \frac{\mathrm{d}^2 X^{D-1}}{\mathrm{d}(x^{D-1})^2} = \alpha\ , \tag{2.4}$$

[1]This expression is valid for $\zeta \neq 0$ and $\zeta \neq 1/4$. The case $\zeta = 1/4$ is discussed below.

where α is a constant. The most general solution of these equations is

$$\sigma = \frac{\alpha}{2} x_\mu x^\mu + k_\mu x^\mu + \sigma_0 \ , \tag{2.5}$$

where k^μ and σ_0 are integration constants. The remaining step is to solve Eq. (2.1c) for $V(\phi)$.

Notably, the solution for the potential turns out to be a local function of $\phi(x)$ for any nonvanishing value of the parameter $\zeta \neq 1/4$ given by Eq. (1.5); see also Fig. 1.

The integration constants α, k^μ and σ_0 are not completely arbitrary, but they are related to the parameters in the potential by

$$\begin{aligned} \lambda_1 &= k_\mu k^\mu - 2\alpha\sigma_0 \ , \\ \lambda_2 &= \alpha \ . \end{aligned} \tag{2.6}$$

Note that if the solution had been assumed to be invariant under translations in some direction, α could only be zero. These solutions exist if the second term in the potential (1.5) vanishes, which only occurs for the special cases $\lambda_2 = 0$ or $\zeta = \zeta_D$.

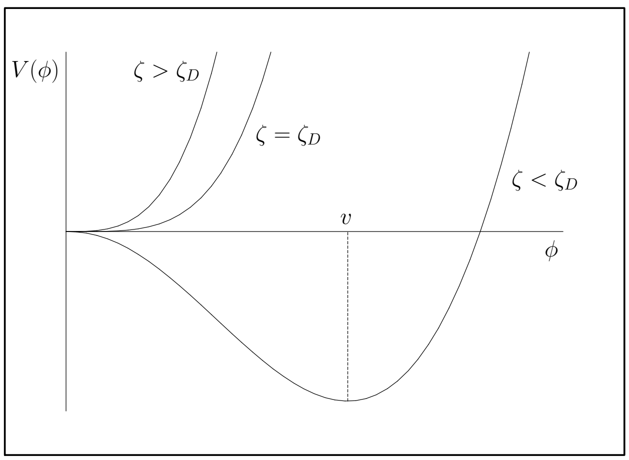

Figure 1: For $0 < \zeta < 1/4$, $\lambda_1 > 0$, and $\lambda_2 > 0$, the potential has a global minimum at $\phi = 0$ if $\zeta \geq \zeta_D$. For $0 < \zeta < \zeta_D$, it has a local maximum at $\phi = 0$ and a global minimum at $v = \left[4(D-1)\lambda_2{\lambda_1}^{-1}(\zeta - \zeta_D)/(2\zeta - 1)\right]^{2\zeta/(1-4\zeta)}$. The critical value for symmetry breaking occurs at the conformal coupling $\zeta = \zeta_D$. For $\zeta = 1/4$ this potential describes again a symmetry breaking self-interaction with the same qualitative features, where the minimum is now at the vacuum expectation value $v = \exp[1/2(-\lambda_1/\lambda_2 - D + 1)]$.

2.1 Generic cases

For the class of potentials of the form (1.5) with $\zeta \neq \zeta_D$, the solutions have no nontrivial integration constants and they are completely determined by the coupling constants of the action, λ_1, λ_2 and ζ. Depending on the value of λ_2, the solutions fall into two excluding cases: Lorentz invariant solutions, including instantons, if $\lambda_2 \neq 0$, and plane-fronted shock waves for $\lambda_2 = 0$.

• $\lambda_2 \neq 0$. After a suitable shift in the coordinates, the solution reads

$$\phi = \left[\frac{\lambda_2}{2} x^\mu x_\mu - \frac{\lambda_1}{2\lambda_2}\right]^{-2\zeta/(1-4\zeta)} . \tag{2.7}$$

This solution describes a spherical shock wave, and the scalar field is singular at $x_\mu x^\mu = \lambda_1/\lambda_2^2$. In the Euclidean continuation in spherical coordinates ($ds^2 = d\rho^2 + \rho^2 d\Omega_{D-1}^2$), the solution is

$$\widehat{\phi} = \left(\frac{\lambda_2}{2}\right)^{-2\zeta/(1-4\zeta)} \left[\rho^2 - \frac{\lambda_1}{\lambda_2^2}\right]^{-2\zeta/(1-4\zeta)} , \tag{2.8}$$

so that the singularity at the origin is not present if $\lambda_1 < 0$. This solution describes an instanton and has finite action for $(D-2)/4D < \zeta < 1/4$,

$$\begin{aligned} I_E[\widehat{\phi}] &= -\int \mathrm{d}^D x \sqrt{g} \left(\frac{1}{2}\widehat{\phi} V'(\widehat{\phi}) - V(\widehat{\phi})\right) , \\ &= 2^{\frac{2-4\zeta}{1-4\zeta}} \Omega^{D-1} \, \frac{\zeta^2}{1-4\zeta} \, (-\lambda_1)^{\frac{D}{2}-\frac{1}{1-4\zeta}} \, \lambda_2^{4\frac{D-\zeta}{1-4\zeta}(\zeta-\zeta_D)} \, B\left[\frac{D}{2}, \frac{1}{1-4\zeta} - \frac{D}{2}\right] , \end{aligned}$$

where Ω^{D-1} is the volume of the sphere S^{D-1}, and $B[m,n] = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ is the Beta function.

• $\lambda_2 = 0$. In this case one obtains

$$\phi = (k_\mu x^\mu)^{-2\zeta/(1-4\zeta)} , \tag{2.9}$$

where $k_\mu k^\mu = \lambda_1$. This solution describes a plane-fronted shock wave traveling along k_μ. Note that although the potential (1.5) does not have a mass term proportional to ϕ^2, this solution has a dispersion relation corresponding to a free particle whose mass is determined by the self coupling constant, $m^2 = -\lambda_1$. Hence the shock wave is tachyonic for positive λ_1.

2.2 Special cases

• $\zeta = \zeta_D$. The conformally coupled case is particularly interesting. The potential has a single coupling constant and acquires the form given by Eq. (1.6), which makes the matter piece of the action (1.1) conformally invariant. The solution is regular at infinity since $1/8 \leq \zeta_D < 1/4$, for $D \geq 3$. Note that in the generic case discussed above, the solution has no integration constants. However for $\zeta = \zeta_D$, the parameter λ_2 is no longer a coupling constant because it drops out from the potential. Hence, in the conformally coupled case, $\lambda_2 = \alpha$ becomes an integration constant and the solution (2.7) acquires a tunable parameter as expected from scale invariance. As a consequence, both types of solutions discussed above can be present now. For $\alpha \neq 0$, the solution is given by (2.7) with $\zeta = \zeta_D$,

$$\phi = \left[\frac{\alpha}{2} x^\mu x_\mu - \frac{\lambda_1}{2\alpha}\right]^{-(D-2)/2} , \tag{2.10}$$

which is regular for $\lambda_1 < 0$. The value of the Euclidean action is finite and given by

$$I_E = (-\lambda_1)^{1-\frac{D}{2}} \frac{(D-2)\sqrt{\pi}\Omega^{D-1}\Gamma\left(\frac{D}{2}\right)}{4\Gamma\left(\frac{D+1}{2}\right)} \ , \tag{2.11}$$

which is independent from the integration constant α. For $\alpha = 0$, the plane-fronted shock wave solution is obtained from (2.9).

• $\zeta = \mathbf{1/4}$. In this case, it is useful to define $\sigma = \ln(\phi)$ and proceed as before. Thus, if $\lambda_2 \neq 0$, the scalar field is given by

$$\phi = \exp\left(\frac{\lambda_2}{2} x^\mu x_\mu - \frac{\lambda_1}{2\lambda_2}\right) , \tag{2.12}$$

and the self-interaction potential reads

$$V(\phi) = \frac{\lambda_2}{2}\phi^2 \left[2\ln\phi + \frac{\lambda_1}{\lambda_2} + D - 1\right] . \tag{2.13}$$

As in the generic case, the solution has no nontrivial integration constants.

• $\zeta = \mathbf{1/4}$ **and** $\boldsymbol{\lambda_2 = 0}$. In this case the potential corresponds to a massive term, $V(\phi) = \frac{1}{2}\lambda_1\phi^2$ $(\lambda_1 > 0)$, and the scalar field is given by $\phi = \exp(k^\mu x_\mu)$ with $k_\mu k^\mu = \lambda_1$. This means that the improvement has the interesting effect of making the tachyonic solutions of the linear Klein-Gordon equation to have vanishing stress energy and hence, devoid of any conserved charge.

Note that a real oscillatory solution of the free massive Klein-Gordon equation in flat spacetime has a non-constant $T_{\mu\nu}$ whereas the improved $\Theta_{\mu\nu}$ with $\zeta = 1/4$ corresponds to dust with constant energy density.

3. Summary and discussion

It has been shown that flat spacetime in D dimensions can be consistently endowed with nontrivial real scalar fields solving the nonlinear Klein-Gordon equation without generating back reaction. This is achieved by modifying the conventional energy-momentum tensor with the addition of a symmetric tensor that is identically conserved, and which depends on the arbitrary parameter ζ. This modification can be seen to result from a nonminimal coupling of the scalar field to gravity: even though this interaction disappears in a flat background, it generates nontrivial effects through the additional term in the stress-energy of matter.

It is observed that requiring the existence of nontrivial solutions with vanishing improved energy-momentum tensor, fixes the self-interaction potential $V(\phi)$ as a local function of the scalar field, with two arbitrary coupling constants. The potential exhibits a symmetry breaking transition at the value of $\zeta = \zeta_D$, for which the matter lagrangian becomes conformally invariant. In this case, the potential has a single coupling constant and the solution acquires a tunable parameter.

The space of allowed solutions is enlarged since the improved energy-momentum tensor permits the use of weaker asymptotic conditions. This is due to the fact that divergences arising from the conventional energy-momentum tensor can be canceled by the contribution of the improvement. This leads to new allowed configurations having finite charges. In fact, the solutions presented here have unbounded standard $T_{\mu\nu}$. The stability of these configurations could be tackled along the lines of Ref. [4].

The solutions couple to the gravitational field without curving spacetime. They describe shock waves, and instantons in the Euclidean continuation. For the free massive Klein-Gordon equation the solutions exist only if $\zeta = 1/4$. These are the tachyons which become part of the spectrum of the (degenerate) vacuum.

The same results found here would occur in any metric gravitation theory that admits flat spacetime as a solution. The essential feature is the nonminimal coupling of the scalar field, while the Einstein-Hilbert term could be replaced by a more general Lagrangian.

In three dimensions, static and time-dependent configurations on nontrivial spacetimes without back reaction are known to exist for scalar fields nonminimally coupled to gravity with negative cosmological constant [5, 6, 7, 8, 9]. We have shown here that this effect is neither due to the particular dimensionality nor to the negative value of the cosmological constant. It is desirable then to explore whether this phenomenon may also occur for nontrivial spacetimes in higher dimensions, as well as in the presence of a non vanishing cosmological constant.

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